NATIONAL ADVISORY COMMITTEE
FOR AERONAUTICS

TECHNICAL NOTE 2917

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TURBULENT BOUNDARY LAYER ON A FLAT PLATE

By Morris W. Rubesin
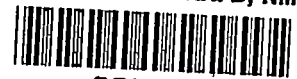
Ames Aeronautical Laboratory
Moffett Field, Calif.

NACA

Washington

March 1953

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SUMMARY

A modified Reynolds analogy is developed for the compressible turbulent boundary layer on a flat plate. When mixing-length theories are used to evaluate terms of the final expressions, it is found for air that the ratio of Stanton number to half the local skin-friction coefficient is greater than unity. At Mach number equals zero, this ratio is of the order of 1.18 to 1.21 for Reynolds numbers based on momentum thickness of 10^3 to 10^6. Up to a Mach number of 5 and under extreme conditions of surface temperature, it is found that the ratio of Stanton number to half the skin-friction coefficient differs from its values for the incompressible case (M=0) by amounts so small as to be of the magnitude of the uncertainties in the theory.

INTRODUCTION

There are several theories in the literature of aerodynamics which are concerned with the subjects of skin friction and heat transfer in the compressible turbulent boundary layer on a flat plate (refs. 1 through 8). Each of these theories, however, is restricted through the assumptions that Prandtl number (Pr) is unity in the laminar sublayer and that there is an equivalence in the mechanisms of the transport of heat and momentum in the turbulent region of the boundary layer.[1] The latter can be considered equal to the assumption that the turbulent Prandtl number (α) is unity. From these assumptions and from a definition of the heat-transfer coefficient based on the temperature difference between that of the surface and the stagnation temperature of the free stream, it is found that there is an exact equivalence between the local heat-transfer coefficient (written in dimensionless fashion as the

[1] The recent mixing-length theory of Lin and Shen (refs. 6, 7, and 8) considers the differences in the turbulent exchange mechanisms of momentum and energy. The theory, however, is incomplete in that it requires empirical knowledge of three coefficients which have not yet been determined.

Stanton number) and the local skin-friction coefficient. This corresponds to the well-known Reynolds analogy of the low-speed case.

From investigations of heat transfer at subsonic speeds, it is known, for the case of air, that the Reynolds analogy underestimates the value of the Stanton number by approximately 20 percent. The results from these investigations, including the empirical work of Colburn (ref. 9), and the results of this analysis are shown in table I (refs. 10 through 14). The analyses in the table are classified according to (1) which of the usual three boundary-layer subdivisions are used, (2) what assumptions are employed in each, (3) what value of turbulent Prandtl number is employed, and (4) whether frictional dissipation is included. In general, the following is observed: Including the buffer layer introduces to the final expressions an additional term which depends on the assumed velocity distribution in that region; including dissipation leads to the evaluation of a temperature recovery factor; and allowing for variation of the turbulent Prandtl number α results in terms of Pr/α where Pr appeared for the cases where $\alpha = 1$. It should be emphasized that only Shirokow, Smith and Harrop, and the present analysis consider the effect of compressibility. The other analyses neglect compressibility and employ low-speed values for the velocity ratios at the outer edges of the laminar sublayer and buffer layer, respectively, and use empirically determined incompressible velocity distributions in evaluating the effect of the buffer layer. The latter procedure introduces the constants which appear in the end expressions developed by von Kármán and Seban. It should be noted that Shirokow also restricts his end results to the incompressible case by employing a low-speed value for the velocity ratio at the outer edge of the sublayer.

The question arises as to how the relative thickening of the sublayer and buffer layer under the conditions of compressibility will influence the results of these analyses. Will compressibility cause the large corrections to the Reynolds analogy to increase or decrease? Smith and Harrop (ref. 14) have examined this problem from the compressibility viewpoint. The identical assumptions used by von Kármán were employed except that all temperatures were replaced by total temperatures to account for frictional dissipation. This procedure is justified whenever the laminar or turbulent Prandtl number is unity, as Smith and Harrop assumed for the turbulent portion of the boundary layer. Since the analysis considered the laminar Prandtl number other than unity in the sublayer and buffer layer, some error was introduced. The main effect of this error was that the recovery temperature of the surface remained at the stagnation temperature whatever the value of Pr. In general, when it was assumed that the surface temperature largely governs the extent of the sublayer and buffer layer, it was found that compressibility has a very small effect on the relationship between heat transfer and skin friction.

In view of the failure of the Smith and Harrop theory to show any variation of temperature recovery factor with Prandtl number and of the omission of the effect of the turbulent Prandtl number, it is believed desirable to re-examine the problem considering the effects imposed by compressibility. It is the purpose of this report, therefore, to study the relationship between heat transfer and skin friction in the compressible turbulent boundary layer on a flat plate with emphasis on the following:

1. The effect of the relative thickening of the sublayer of the boundary layer
2. The effect of frictional dissipation
3. The effect of differences in the turbulent exchange mechanism of momentum and heat (turbulent Prandtl numbers other than unity)

Because of the present lack of understanding of the mechanism of turbulence in shear flow and of the structure of compressible turbulent boundary layers, this theory, like all the other theories of the turbulent boundary layer, depends largely on arbitrary, but plausible, assumptions. It cannot, therefore, be considered as absolute.

SYMBOLS

c_f local skin-friction coefficient, $\dfrac{\tau}{(1/2)\rho_\infty u_\infty^2}$

c_p specific heat at constant pressure

E total energy, $c_p T + u^2/2$

h local heat-transfer coefficient, rate of heat transfer per unit area per degree, $q/(T_w - T_{aw})$

H thermal energy flux (defined by eq. (8))

k thermal conductivity

M Mach number

p pressure

Pr Prandtl number composed of properties based on molecular transport, $\mu c_p/k$

q rate of heat transfer per unit area

R_θ Reynolds number based on momentum thickness

r   local recovery factor, $T_{aw}-T_\infty/T_S-T_\infty$

St   Stanton number, $h/\rho_\infty u_\infty c_{p\infty}$

T   temperature

u   velocity in the x direction

v   velocity in the y direction

x,y   coordinate system with $y = 0$ plane as surface of the plate

α Prandtl number composed of properties based on turbulent transport, $\epsilon c_p/\kappa$

γ ratio of specific heats (1.4 for air)

ϵ eddy viscosity (defined by eq. (9))

θ momentum thickness

κ eddy thermal conductivity (defined by eq. (10))

μ viscosity

ρ density

τ shear stress (defined by eq. (7))

Superscripts

—   mean value

'   fluctuating value

Subscripts

1   condition at outer edge of sublayer

2   condition at outer edge of buffer layer

S   stagnation conditions

∞ condition at outer edge of boundary layer

w   condition of surface

aw   condition at surface for zero heat transfer

System of units used is arbitrary but must be selfconsistent.

ANALYSIS

The basic compressible turbulent boundary-layer equations are given in reference 5 and can be written for steady flow past a flat plate as

$$\frac{\partial}{\partial x}(\bar{\rho}\bar{u}^2) + \frac{\partial}{\partial y}(\bar{\rho}\bar{u}\bar{v}) = \frac{\partial}{\partial y}(\bar{u}\frac{\partial\bar{u}}{\partial y} - \bar{u}\overline{\rho'v'} - \bar{\rho}\cdot\overline{u'v'}) \qquad (1)$$

$$\frac{\partial}{\partial x}(\bar{\rho}\bar{u}\bar{E}) + \frac{\partial}{\partial y}(\bar{\rho}\bar{v}\bar{E}) = \frac{\partial}{\partial y}(\bar{\mu}\frac{\partial\bar{E}}{\partial y} - \bar{\rho}\overline{E'v'} - \bar{E}\overline{\rho'v'}) + \left(\frac{1}{Pr} - 1\right)\frac{\partial}{\partial y}(\bar{\mu}\frac{\partial}{\partial y}c_p\bar{T}) \qquad (2)$$

$$\frac{\partial}{\partial x}(\bar{\rho}\bar{u}) + \frac{\partial}{\partial y}(\bar{\rho}\bar{v} + \overline{\rho'v'}) = 0 \qquad (3)$$

At present, equations (1) to (3) cannot be solved rigorously. They can only be used as guides toward choosing the important variables and for indicating the form of simplified, though arbitrary, relationships between these variables.

For choosing the variables to be considered, equations (1) to (3) are rewritten using the relation

$$E = c_p T + \frac{1}{2}u^2 \qquad (4)$$

as

$$\bar{\rho}\bar{u}\frac{\partial\bar{u}}{\partial x} + (\bar{\rho}\bar{v} + \overline{\rho'v'})\frac{\partial\bar{u}}{\partial y} = \frac{\partial}{\partial y}(\bar{\mu}\frac{\partial\bar{u}}{\partial y} - \bar{\rho}\overline{u'v'}) \qquad (5)$$

$$\bar{\rho}\bar{u}\frac{\partial}{\partial x}(c_p\bar{T} + \frac{\bar{u}^2}{2}) + (\bar{\rho}\bar{v} + \overline{\rho'v'})\frac{\partial}{\partial y}(c_p\bar{T} + \frac{\bar{u}^2}{2}) =$$

$$\frac{\partial}{\partial y}(\bar{k}\frac{\partial\bar{T}}{\partial y} + \bar{u}\bar{\mu}\frac{\partial\bar{u}}{\partial y} - \bar{\rho}c_p\overline{v'T'} - \bar{u}\bar{\rho}\overline{u'v'}) \qquad (6)$$

In equations (5) and (6), the fluctuating density terms have been placed on the left side of the equations. By virtue of the continuity equation (3), it is plausible to consider the fluctuating density terms as virtual masses which contribute to those terms which represent the change of momentum or of total energy of a unit volume of fluid at a point. As a result of this, the fluctuating density terms do not appear in the terms which are considered to represent the transport of momentum and of thermal energy on the right of the equations.

The terms within the operators on the right side of equations (5) and (6), called the shear stress and energy flux, are defined as

$$\tau = \bar{\mu} \frac{\partial \bar{u}}{\partial y} - \bar{\rho} \ \overline{u'v'} = (\mu + \epsilon) \frac{\partial u}{\partial y} \tag{7}$$

$$H = \bar{k} \frac{\partial \bar{T}}{\partial y} - \bar{\rho} \ c_p \ \overline{v'T'} + \bar{u}\bar{\mu} \frac{\partial \bar{u}}{\partial y} - \bar{u}\bar{\rho} \ \overline{u'v'} \tag{8}$$

$$= (k + \kappa) \frac{\partial T}{\partial y} + u(\mu + \epsilon) \frac{\partial u}{\partial y}$$

where the eddy viscosity ϵ is defined as

$$\epsilon = - \frac{\bar{\rho} \ \overline{u'v'}}{\partial \bar{u}/\partial y} \tag{9}$$

and the eddy thermal conductivity κ is defined as

$$\kappa = - \frac{\bar{\rho} \ c_p \ \overline{v'T'}}{\partial \bar{T}/\partial y} \tag{10}$$

The bars have been dropped in the final terms of equations (7) and (8) because all the terms represent mean values.

Although equations (5) and (6) cannot be solved rigorously, they do yield sufficient information to act as a guide in treating the problem approximately. For the case where $Pr = 1$ and $\alpha = 1$, it can be shown that the dependent variables in equations (5) and (6) are linearly related to each other, that is,

$$c_p T + \frac{u^2}{2} = Au + B$$

where A and B are constants. This is equivalent to expressing H/τ equal to a constant through the boundary layer. For the case where both Pr and α differ from but are close to unity, it is plausible to assume, as a first approximation, that H/τ still remains constant through the boundary layer, though the constant may be dependent

on Pr and a. It should be noted that this assumption, that H/τ = constant, is the basis for the present theory.

The ratio of the terms of equation (8) to those of equation (7) is

$$\frac{H}{\tau} = \frac{k + \kappa}{\mu + \epsilon} \left.\frac{\partial T}{\partial u}\right|_x + u \tag{11}$$

The relation between temperature and velocity in the boundary layer is obtained on integration of equation (11)

$$T - T_w = \int_0^u \frac{\mu + \epsilon}{k + \kappa} \left(\frac{H}{\tau} - u\right) du \tag{12}$$

At the surface of the plate, the molecular transport terms predominate and equations (7) and (8) become

$$\tau = \left(\mu \frac{\partial u}{\partial y}\right)_w = \tau_w$$

$$H = \left(k \frac{\partial T}{\partial y}\right)_w = - q_w$$

Therefore,

$$\frac{H}{\tau} = - \frac{q_w}{\tau_w} \tag{13}$$

Equation (12) becomes

$$T_w - T = \int_0^u \frac{\mu + \epsilon}{k + \kappa} \left(\frac{q_w}{\tau_w} + u\right) du \tag{14}$$

Equation (14) can be rewritten as

$$\frac{q_w}{\tau_w} = \frac{T_w - T - \int_0^u \frac{\mu + \epsilon}{k + \kappa} u \, du}{\int_0^u \frac{\mu + \epsilon}{k + \kappa} \, du} \tag{15}$$

For the case of zero convective heat transfer at the surface, the numerator of equation (15) must be zero, requiring the surface temperature to be given by

$$T_w = T_{aw} = T_\infty \left(1 + \frac{1}{T_\infty} \int_0^{u_\infty} \frac{\mu + \epsilon}{k + \kappa} u \, du\right) \tag{16}$$

or

$$T_{aw} = T_\infty \left[1 + \frac{\gamma - 1}{2} M_\infty{}^2 \int_0^1 c_{p\infty} \frac{\mu + \epsilon}{k + \kappa} \, d\left(\frac{u}{u_\infty}\right)^2 \right] \tag{17}$$

The quantity T_{aw} is called the recovery temperature. When the heat-transfer coefficient is defined as

$$h = \frac{q_W}{T_W - T_{aw}} \tag{18}$$

the ratio of Stanton number to skin-friction coefficient becomes

$$\frac{St}{\frac{c_f}{2}} = \frac{\dfrac{h}{\rho_\infty \mu_\infty c_{p\infty}}}{\dfrac{c_f}{2}} = \frac{1}{\displaystyle\int_0^1 \frac{c_{p\infty}(\mu + \epsilon)}{k + \kappa} \, d\left(\frac{u}{u_\infty}\right)} \tag{19}$$

From equation (16) and the definition of recovery factor

$$r = \frac{T_{aw} - T_\infty}{T_S - T_\infty} \tag{20}$$

the expression for recovery factor is

$$r = \int_0^1 \frac{c_{p\infty}(\mu + \epsilon)}{k + \kappa} \, d\left(\frac{u}{u_\infty}\right)^2 \tag{21}$$

Equations (19) and (21) constitute the general expressions for the ratio $St/(c_f/2)$ and the recovery factor r. Integration of these equations is facilitated by the assumption that the boundary layer can be divided into three parts: the sublayer, where only the molecular transport terms appear (integrand = Pr); the turbulent portion, where only the eddy transport terms appear (integrand = α); and the buffer layer, where both kinds of transport occur, the integrand depending on the proportion of each type of transport mechanism. To establish the proportion of turbulent to molecular exchange in the buffer layer, von Kármán, Reichardt, and Seban used velocity distributions to establish ϵ and then computed κ by making an assumption of the value of α. (See table I.) Smith and Harrop carried this over to the compressible case by assuming that the velocity distribution in the buffer layer was identical to that used by von Kármán when the fluid properties are evaluated at the wall temperature and when the velocity term is arbitrarily replaced by a function of velocity obtained from their expressions for the fully turbulent-flow region. Because of the uncertainty of the latter procedure, the equally uncertain but simpler approach of entirely neglecting the buffer region and somewhat thickening the sublayer will be used in this analysis. It can be deduced from

table I by comparing the Prandtl-Taylor theory with von Kármán's theory, for the case of $0.5 < \text{Pr} < 2$, that a very small error is introduced by omitting the buffer region and replacing it by a thicker sublayer.

When the integrand in equations (19) and (21) is set equal to Pr in the sublayer, and α in the turbulent portion, the equations become

$$\frac{\text{St}}{\frac{c_f}{2}} = \frac{1}{\alpha \left[1 - \left(1 - \frac{\text{Pr}}{\alpha} \right) \frac{u_1}{u_\infty} \right]} \tag{22}$$

and

$$r = \alpha \left[1 - \left(1 - \frac{\text{Pr}}{\alpha} \right) \left(\frac{u_1}{u_\infty} \right)^2 \right] \tag{23}$$

It is observed from table I that for $\alpha = 1$, equation (22) is identical with those derived by Taylor, Prandtl, and Shirokow. For $\alpha = 1$, the recovery-factor expression is identical with Shirokow's.

When α is eliminated between equations (22) and (23) there results

$$\frac{\text{St}}{\frac{c_f}{2}} = \frac{1 + \frac{u_1}{u_\infty}}{r + \text{Pr}\,\frac{u_1}{u_\infty}} \tag{24}$$

DISCUSSION

The results of this analysis have been shown to depend on a knowledge of the value of the velocity ratio at the outer edge of the sublayer and of the value of the turbulent Prandtl number α. Neither of these quantities can be expressed with certainty; however, it will be shown that sufficient information is available to allow numerical evaluation of the ratio of Stanton number to the local skin-friction coefficient.

There are alternative procedures for the evaluation of u_1/u_∞. The most widely used method is to assume that low-speed expressions prevail under the conditions of compressibility with the exception that the fluid properties are evaluated at the surface temperature. Accordingly, Frankl and Voishel (ref. 1) express this as

$$\frac{u_1}{u_\infty} = 11.5 \sqrt{\frac{c_f}{2}} \sqrt{\frac{T_w}{T_\infty}} \tag{25}$$

The use of this relationship in Frankl and Voishel's theory for skin friction (also implied in Van Driest's theory) is known to yield skin-friction coefficients which agree well with available experimental data at Mach numbers up to 2.5 (ref. 15). Equation (25) will be used in the numerical evaluations which follow; however, effects of the uncertainties introduced will be noted.

In order to use equation (25), it is necessary to be able to evaluate the local skin-friction coefficient c_f. At present there are insufficient experimental data to establish c_f under general conditions of surface temperature and Mach number. It is necessary to rely on theory. For this analysis the results of the appendix, in which the well-known Van Driest analysis is repeated for $Pr \neq 1$ and $\alpha \neq 1$, are used in evaluation of c_f.

Present knowledge of the turbulent Prandtl number α is also quite limited. The term α, unlike Pr, is believed to depend on the type of flow as well as on the fluid because differences have been found in low-speed tests of jets and of pipes or channels. In air jets it has been found (ref. 16) that α varies from 0.70 to 0.77. These values of α are over-all values determined from the spreading of the jets. In a pipe it has been found (ref. 17) that the local α varies from 0.82 to 1.06, depending on the radial position and on the Reynolds number. Experiments in rectangular channels (ref. 18) also indicate a variation of local α ranging from 0.60 to 0.95 due to position and Reynolds number. There are no data available, however, for the magnitude of α in boundary layers on a flat plate.

In view of the arbitrary character of the assumption concerning the velocity ratio at the outer edge of the sublayer and of the uncertainty of the value of α, can the results of this theory yield information of any accuracy? It is believed that equation (24) allows a fairly accurate estimation of the ratio $St/(c_f/2)$. The reasoning which leads to this conclusion is based on the considerations of the following paragraph.

When a constant value of α, having the same order of magnitude as is measured subsonically in pipes, and the value of u_1/u_∞ based on equation (25) are introduced into equation (23), it is found that the recovery factor is lowered from its incompressible value by as much as 10 percent at a Mach number of 4. This is contrary to experimental experience where the recovery factor r remains essentially constant with Mach number in the range $0 < M < 3.8$ (refs. 19, 20, 21, and 22). To conform with experiment, α or u_1/u_∞ or both would have to be altered as a function of Mach number to maintain the recovery factor constant. To avoid this speculation, the direct influence of α is eliminated through the use of equation (24) where the experimentally verified expression $r = Pr^{1/3}$ is used. The primary source of error in equation (24), then, is the uncertainties in the value of u_1/u_∞. Because of the nature of equation (24) when Pr is near 0.7, its

approximate value for gases, uncertainties in u_1/u_∞ are not too serious. On the average, in the range $0 < M < 5$ a 20-percent error in u_1/u_∞ represents a 1-percent error in the ratio $St/(c_f/2)$.

Values of $St/(c_f/2)$ determined from equation (24) are shown in figures 1 and 2 for $Pr = 0.72$. The two limiting surface temperatures likely to be encountered are shown, respectively, in the two figures. These are the recovery temperature and the free-stream temperature. For the case of $M = 0$ both surface temperatures are equal, and the results of figures 1 and 2 indicate that the ratio of the Stanton number to half the local skin-friction coefficient is from about 17.5 percent to 20.5 percent higher than the Reynolds analogy. Colburn's (ref. 9) empirical correction to Reynolds analogy is $Pr^{-2/3}$ and corresponds to a 25-percent correction. The empirical correction, however, is based on data below a length Reynolds number of one million where the correction of this theory would be about 21 percent. This correspondence of theory and experiment is well within the scatter of the experimental data. For the case of a surface at very near the recovery temperature, it is noted from figure 1 that the influence of Mach number is small, increasing the ratio of the Stanton number to half the local skin-friction coefficient at $M = 5$ by only about 3 percent over the incompressible case. For the cooled case the effect of Mach number is again small, lowering the ratio $St/(c_f/2)$ by about 1 percent at $M = 5$. These variations can be considered to be the same order of magnitude as the uncertainties in the theory for Prandtl number near unity.

A comparison of the results of the present theory with the results of Smith and Harrop for $R_\theta = 10^5$ is also shown in figure 1. It is observed that the Smith and Harrop theory yields results which are markedly lower than the results of the present theory. By varying the reference velocity for the buffer layer (u_0 in table I) between its limits, it was found that the Smith and Harrop values varied by less than 3 percent. The main difference in the results of the two theories, therefore, is due to the consideration of recovery factor in the present theory. A similar comparison to that shown in figure 1 cannot be made in figure 2. For the condition $T_W = T_\infty$, it is found that the series in the denominator in Smith and Harrop's end equation (table I) does not converge rapidly enough to give a valid answer at Mach numbers of the order of unity or higher.

CONCLUDING REMARKS

From a simple analysis of the effect of compressibility on the relationship between heat transfer and skin friction for air, a modified Reynolds analogy, it is found that the ratio of Stanton number to half the local skin-friction coefficient at $M = 0$ is from 17.5 percent to 20.5 percent higher than given by Reynolds analogy in the range of momentum-thickness Reynolds number from 10^3 to 10^6.

It is also found that up to a Mach number of 5 and under extreme conditions of surface temperatures, the effects of compressibility on this ratio are so small as to be of the same order of magnitude as the possible uncertainties of the theory.

Ames Aeronautical Laboratory,
        National Advisory Committee for Aeronautics,
                Moffett Field, Calif., Dec. 8, 1952.

APPENDIX

SKIN-FRICTION ANALYSIS

In this appendix, the Van Driest analysis will be repeated with the modifications that $Pr \neq 1$ and $\alpha \neq 1$.

The basic equation for determining the velocity distribution in the turbulent portion is

$$\tau = \rho K^2 y^2 \left(\frac{du}{dy}\right)^2 \tag{A1}$$

where the quantities represented are temporal mean values.

By algebraic manipulation, equation (A1) is transformed to

$$\frac{du}{\sqrt{\rho_W/\rho}} = \frac{1}{K}\sqrt{\frac{\tau_W}{\rho_W}}\,\frac{dy}{y} \tag{A2}$$

Equation (A2) corresponds to equation (48) of reference 2. Letting $\tilde{u} = u/u_\infty$, equation (A2) becomes

$$\frac{d\tilde{u}}{\sqrt{\rho_W/\rho}} = \frac{1}{Ku_\infty}\sqrt{\frac{\tau_W}{\rho_W}}\,\frac{dy}{y} \tag{A3}$$

Since the static pressure across the boundary layer is considered constant

$$\frac{\rho_W}{\rho} = \frac{T}{T_W} \tag{A4}$$

On integration of equation (12) of the text, the right member of equation (A4) can be written as

$$\frac{T}{T_W} = b^2 \left(1 + B\tilde{u} - A^2 \tilde{u}^2\right) \tag{A5}$$

where

$$b^2 = 1 - m\beta \left(1 - \frac{Pr}{\alpha}\right) + n^2\beta^2\left(1 - \frac{Pr}{\alpha}\right)$$

$$\beta = \frac{u_1}{u_\infty}$$

$$m = \frac{1 + \frac{\gamma-1}{2}\,\alpha\,M_\infty^2\left[1 - \left(1 - \frac{Pr}{\alpha}\right)\beta^2\right] - \frac{T_w}{T_\infty}}{\frac{T_w}{T_\infty}\left[1 - \left(1 - \frac{Pr}{\alpha}\right)\beta\right]} \qquad (A6)$$

$$n^2 = \frac{\frac{\gamma-1}{2}\,\alpha\,M_\infty^2}{\frac{T_w}{T_\infty}}$$

$$B = \frac{m}{b^2}$$

$$A^2 = \frac{n^2}{b^2}$$

When $\alpha = Pr = 1$, equations (A5) and (A6) degenerate to those used by Van Driest. Substitution of equation (A5) into equation (A3) results in

$$\frac{d\tilde{u}}{\sqrt{1 + B\tilde{u} - A^2\tilde{u}^2}} = \frac{b}{Ku_\infty}\sqrt{\frac{\tau_w}{\rho_w}}\frac{dy}{y} \qquad (A7)$$

Except for the term b, equation (A7) is identical with equation (52) of Van Driest. Following Van Driest, equation (A7) is integrated to yield

$$\frac{1}{A}\sin^{-1}\frac{2A^2\tilde{u} - B}{(B^2 + 4A^2)^{1/2}} + \frac{1}{A}\sin^{-1}\frac{B}{(B^2 + 4A^2)^{1/2}} =$$

$$\frac{1}{u_\infty}\sqrt{\frac{\tau_w}{\rho_w}}\left(F + \frac{b}{K}\ln\sqrt{\frac{\tau_w}{\rho_w}}\frac{y}{\nu_w}\right) \qquad (A8)$$

where F is a constant.

When equations (A4), (A5), and (A8) are substituted into

$$\tau_w = \frac{d}{dx}\int_0^\delta \rho u\,(u_\infty - u)\,dy \qquad (A9)$$

there results

$$\tau_w = \frac{u_\infty\mu_w}{K}\frac{d}{dx}\left[\frac{D}{b}\,a^2 J e^{\frac{a}{A}\sin^{-1}\frac{B}{(B^2 + 4A^2)^{1/2}}}\right] \qquad (A10)$$

where

$$D = e^{-\frac{FK}{b}}$$

$$a = \frac{Ku_\infty}{b\sqrt{\tau_W/\rho_W}}$$

$$J = \int_0^1 \frac{\tilde{u}(1 - \tilde{u})}{(1 + B\tilde{u} - A^2\tilde{u}^2)^{3/2}} \; e^{\frac{a}{A}\sin^{-1}\frac{2A^2\tilde{u} - B}{(B^2 + 4A^2)^{1/2}}} \; d\tilde{u} \tag{A11}$$

For a flat plate

$$\frac{\tau_W}{\rho_\infty u_\infty} = \frac{c_f}{2} = \frac{d\theta}{dx} \tag{A12}$$

Therefore, from equation (A10)

$$\frac{c_f}{2} = \frac{d}{dx}\left[\frac{\mu_W}{K\rho_\infty u_\infty} \frac{D}{b} \; a^2 Je^{\frac{a}{A}\sin^{-1}\frac{B}{(B^2 + 4A^2)^{1/2}}}\right] \tag{A13}$$

or, from equation (A12),

$$\theta = \frac{\mu_W}{K\rho_\infty u_\infty} \frac{D}{b} a^2 Je^{\frac{a}{A}\sin^{-1}\frac{B}{(B^2 + 4A^2)^{1/2}}} \tag{A14}$$

When R_θ is defined as

$$R_\theta = \frac{u_\infty \rho_\infty \theta}{\mu_\infty} \tag{A15}$$

and

$$\frac{\mu_W}{\mu_\infty} = \left(\frac{T_W}{T_\infty}\right)^\omega$$

equation (A14) becomes

$$R_\theta = \left(\frac{T_W}{T_\infty}\right)^\omega \frac{D}{Kb} \; a^2 Je^{\frac{a}{A}\sin^{-1}\frac{B}{(B^2 + 4A^2)^{1/2}}} \tag{A16}$$

Van Driest has shown that a first approximation to J obtained by integration by parts results in

$$a^2 J = \frac{1}{(1 + B - A^2)^{1/2}} \; e^{\frac{a}{A}\sin^{-1}\frac{2A^2 - B}{(B^2 + 4A^2)^{1/2}}} \tag{A17}$$

Thus R_θ can be written

$$R_\theta = \left(\frac{T_W}{T_\infty}\right)^\omega \frac{e^{-\frac{F\kappa}{b}}}{\kappa b} \frac{1}{(1 + B - A^2)^{1/2}} \exp\left[\frac{a}{A}(\sin^{-1}\varphi + \sin^{-1}\psi)\right] \quad (A18)$$

where

$$\varphi = \frac{2A^2 - B}{(B^2 + 4A^2)^{1/2}} \qquad \psi = \frac{B}{(B^2 + 4A^2)^{1/2}}$$

For the case of $M_\infty = 0$ and $T_W/T_\infty = 1$, equation (A18) was integrated to yield a relation between the average skin-friction coefficient and the Reynolds number based on the distance along the plate. Agreement with low-speed experimental results is obtained when $F = 6.5$.

To determine how much the deviation of Pr and α from unity influences the local skin-friction coefficient, the results expressed by equation (A18) are shown in figures 3 and 4. The dashed lines represent the relationship between $c_f/2$ and R_θ when Pr = 0.72 and r = 0.89 (α computed to agree with recovery factor). The solid lines represent the corresponding relationship for Pr = 1, α = 1 and, consequently, r = 1. It is observed that the deviation of Pr and α from unity produces very little effect on $c_f/2$ at the lower Mach numbers. At the higher Mach numbers the effect has some Reynolds number dependence; however, the largest deviation between the two results is of the order of 7 percent. It should be noted that the Van Driest theory is known to yield values of skin-friction coefficient which are higher than measured values on the order of 11 percent at M = 2.5 (ref. 15). The allowed variation of Pr or α does not improve this situation.

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TABLE I.- MODIFIED REYNOLDS ANALOGIES

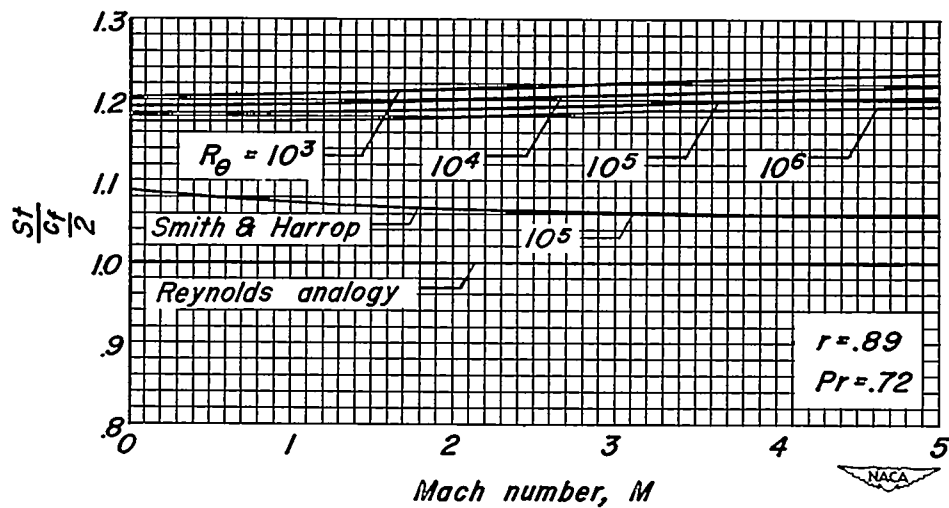| Investigator	Boundary Layer Regions and Governing Assumptions			Turbulent Prandtl Number	Dissipation	Final Expressions
	Laminar Sublayer	Buffer Layer	Turbulent Region			
Reynolds (1874) (Ref. 10)	None	None	$\frac{q}{\tau c_p} = \frac{T - T_l}{u - u_l}$	1	No	Interpreted as $\frac{St}{\left(\frac{c_f}{2}\right)} = 1$
Taylor (1916) Prandtl (1910) (Ref. 10)	$\frac{\tau}{\rho} = $ constant $q = $ constant	None	$\frac{q}{\tau c_p} = \frac{T - T_w}{u - u_w}$	1	No	$\frac{St}{\left(\frac{c_f}{2}\right)} = \frac{1}{1 - (1 - Pr)\frac{u_l}{u_\infty}}$ $\frac{u_l}{u_\infty} = 11.5\sqrt{\frac{c_f}{2}}$
Colburn (1933) (Ref. 9)	←———————— Empirical ————————→					$\frac{St}{\left(\frac{c_f}{2}\right)} = Pr^{-\frac{2}{3}}$
von Kármán (1939) (Ref. 10)	$\frac{\tau}{\rho} = $ constant $q = $ constant	$\frac{\tau}{\rho} = $ constant $q = $ constant Nikuradse velocity distribution	$\frac{q}{\tau c_p} = \frac{T - T_w}{u - u_w}$	1	No	$\frac{St}{\left(\frac{c_f}{2}\right)} = \frac{1}{1 + 5\sqrt{\frac{c_f}{2}}(Pr - 1 + \ln[1 + \frac{5}{6}(Pr - 1)])}$ $\frac{u_l}{u_\infty} = 5\sqrt{\frac{c_f}{2}}$ $\frac{u_2}{u_\infty} = 14\sqrt{\frac{c_f}{2}}$
Reichardt (1940) (Ref. 11)	$\tau = $ constant $q = $ constant	$\frac{q}{\tau} = $ constant Nikuradse, Reichardt velocity distribution	$\frac{q}{\tau} = $ constant	Arbitrary	No	$\frac{St}{\left(\frac{c_f}{2}\right)} = \frac{1}{[1 - (1 - \frac{Pr}{\sigma})\frac{u_l}{u} + (\frac{u_2}{u} - \frac{u_l}{u})(\frac{\ln Pr/\sigma}{1 - \sigma/Pr} - 1)]}$ $\frac{u_l}{u_\infty} = 2\sqrt{\frac{c_f}{2}}$ $\frac{u_2}{u_\infty} = 15\sqrt{\frac{c_f}{2}}$
Shirokow (1936) (Ref. 12)	$\tau = $ constant $u\tau - q = $ constant	None	$\frac{q}{\tau c_p} = \frac{T_l - T_w}{u - u_w}$	1	Yes	$\frac{St}{\left(\frac{c_f}{2}\right)} = \frac{1}{[1 - (1 - Pr)\frac{u_l}{u_\infty}]}$ $r = 1 - (1 - Pr)(\frac{u_l}{u_\infty})^2$ $\frac{u_l}{u_\infty} = \frac{1}{r}$
Seban (1948) (Ref. 13)	$\tau = $ constant $u\tau - q = $ constant	$\tau = $ constant $u\tau - q = $ constant Nikuradse velocity distribution (1) $\frac{u}{\tau} - \frac{q}{\tau} = $ constant (2) $u - \frac{q}{\tau} = $ constant	(1) $T + \frac{u^2}{2c_p} = k_1 u + k_2$ (2) $T + \frac{u^2}{2c_p} = k_3 u + k_4$	Arbitrary	Yes	$\frac{St}{\left(\frac{c_f}{2}\right)} = \frac{1}{[1 + 5\sqrt{\frac{c_f}{2}}(\frac{Pr}{\sigma} - 1 + \ln[1 + \frac{5}{6}(\frac{Pr}{\sigma} - 1)])]}$ (1) $r = 1 - (Y - Pr)(\frac{u_l}{u_\infty})^2 - (1 - Y)(\frac{u_2}{u_\infty})^2 + 2[\frac{u_2}{u_\infty} - (\frac{u_l}{u_\infty})^2](a - 1)$ (2) $r = a - (Y - Pr)(\frac{u_l}{u_\infty})^2 - (a - Y)(\frac{u_2}{u_\infty})^2$ $Y = \frac{1}{\frac{u_2}{u_\infty} - \frac{u_l}{u_\infty}} \int_{u_l}^{u_2} \frac{u + u_l}{u + \sigma} d(\frac{u}{u_\infty})$
Smith and Harrop (1949) (Ref. 14)	$\frac{\tau}{\rho u} = $ constant $\frac{\tau u}{\rho u} - q = $ constant	$\frac{\tau}{\rho u} = $ constant $\frac{\tau u}{\tau + \rho c_p} - q = $ constant	$\frac{q}{\tau c_p} = \frac{T - T_w}{u - u_w}$ or $u\tau - q = $ constant	1	Yes	$\frac{St}{\left(\frac{c_f}{2}\right)} = \frac{1}{1 + 5\sqrt{\frac{c_f}{2}}[Pr - 1 + \ln(\ldots)] - \ldots}$ $A = 1 - \frac{St}{\left(\frac{c_f}{2}\right)}(1 - \frac{T_w}{T_\infty})\frac{u_l}{u_\infty} - \frac{u^2}{M + 15}(\frac{u_l}{u_\infty})^2$ where $5\sqrt{\frac{c_f}{2}}\sqrt{\frac{T_\infty}{T_w}} < \frac{u_l}{u_\infty} < 14\sqrt{\frac{c_f}{2}}\sqrt{\frac{T_\infty}{T_w}} + \ldots$
Rubesin (1953)	$u - \frac{q}{\tau} = $ constant	None	$u - \frac{q}{\tau} = $ constant	Arbitrary	Yes	$\frac{St}{\left(\frac{c_f}{2}\right)} = \frac{1 + \frac{u_l}{u_\infty}}{r + Pr(\frac{u_l}{u_\infty})}$ $\frac{u_l}{u_\infty} = 11.5\sqrt{\frac{c_f}{2}}\sqrt{\frac{T_\infty}{T_w}}$ $r = Pr^{\frac{1}{3}}$

Figure 1.— Modified Reynolds analogy for compressible boundary layer (insulated surface temperature).
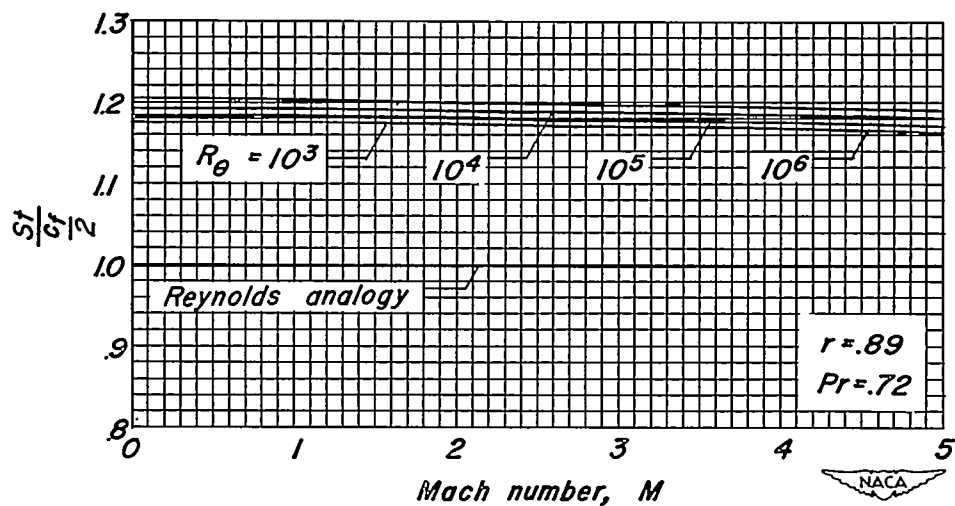

Figure 2.— Modified Reynolds analogy for compressible boundary layer (wall temperature equal to free-stream temperature).